Discrete mathematics
NOTES

B.TECH (SEM-III)

DEPARTMENT OF CSE & CE

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Discrete mathematics

Discrete [individually separate and distinct] mathematics is the branch of mathematics dealing with objects that can consider only distinct, separated values. The term discrete objects include large variety of items such as people, books, and computers and so on.

[UNIT 1] Fundamental Structures and Basic Logic

Set

A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket.

Some Example of Sets

- A set of all positive integers
- A set of all the states in India

Let's say we have a set of Natural Numbers then it will have all the natural numbers as its member and the collection of the numbers is well defined that they are natural numbers. A set is always denoted by a capital letter.

For Example, a set of Natural Numbers is given by

$$N = \{1, 2, 3, 4\ldots\}.$$

Elements of a Set

The objects contained by a set are called the elements of the set. For Example, in the set of Natural Numbers 1, 2, 3, etc are the objects contained by the set, hence they are called the elements of the set of Natural Numbers. We can also say that 1 belongs to set N. It is represented as $1 \in N$, where $\in$ is the symbol of belongs to.

Examples of Sets

Set of Natural Numbers: $N = \{1, 2, 3, 4\ldots\}$
Set of Even Numbers: $E = \{2, 4, 6, 8\ldots\}$
Set of Integers: $Z = \{\ldots, -4, -3, -2, -1, 0, 1, 2,\ldots\}$

Cardinality of a Set

The number of elements present in a set is called the Cardinal Number of a Set. For Example, let's say P is a set of the first five prime numbers given by

$P = \{2, 3, 5, 7, 11\}$, then the Cardinal Number of set P is 5.

The Cardinal Number of Set P is represented by n(P) or |P| = 5.

Cardinality of a set S, denoted by |S|, is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞

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Example − |{1,4,3,5}|=4,|{1,2,3,4,5,...}|=∞

If there are two sets X and Y,

- *$|X|=|Y|$ denotes two sets X and Y having same cardinality. It occurs when the number of elements in X is exactly equal to the number of elements in Y. In this case, there exists a bijective function 'f' from X to Y.*
- *$|X|≤|Y|$ denotes that set X's cardinality is less than or equal to set Y's cardinality. It occurs when number of elements in X is less than or equal to that of Y. Here, there exists an injective function 'f' from X to Y.*
- *$|X|<|Y|$ denotes that set X's cardinality is less than set Y's cardinality. It occurs when number of elements in X is less than that of Y. Here, the function 'f' from X to Y is injective function but not bijective.*
- *If $|X|≤|Y|$ and $|X|≥|Y|$ then $|X|=|Y|$. The sets X and Y are commonly referred as equivalent sets.*

Some Standard Sets used in Set Theory

➢ Set of Natural Numbers is denoted by N
➢ Set of Whole Numbers(defined as positive integers including zero) is denoted by W
➢ Set of Integers is denoted by Z
➢ Set of Rational Numbers(A fraction with non-zero denominators) is denoted by Q
➢ Set of Irrational Numbers(numbers cannot be expressed as the ratio of two integers) is denoted by P
➢ Set of Real Numbers is denoted by R

Representation of Sets

Sets are primarily represented in two forms

1) Roster Form

In the Roster Form of the set, the elements are placed inside braces {} and are separated by commas. Let's say we have a set of the first five prime numbers then it will be represented by

P = {2, 3, 5, 7, 11}.

Here the set P is an example of a finite set as the number of elements is finite, however, we can come across a set that has infinite elements then in that case the roster form is represented in the manner that some elements are placed followed by dots to represent infinity inside the braces.

Let's say we have to represent a set of Natural Numbers in Roster Form then its Roster Form is given as N = {1, 2, 3, 4…..}.

2) Set Builder Form

In Set Builder Form, a rule or a statement describing the common characteristics of all the elements is written instead of writing the elements directly inside the braces.

For Example, a set of all the prime numbers less than or equal to 10 is given as

P = {p : p is a prime number ≤ 10}.

In another example, the set of Natural Numbers in set builder form is given as

N = {n : n is a natural number}.

Types of Sets

There are different types of sets categorized on various parameters. The different types of sets are explained below:

Empty Set

A set that has no elements inside it is called an Empty Set. It is represented by Φ or {}.
For Example   A = {x : x ∈ N and 2 < x < 3}.
Here, between 2 and 3, no natural number exists, hence A is an Empty Set. Empty Sets are also known as Null Sets.

Singleton Set

A set that has only one element inside it is called a Singleton Set.
For Example, B = {x : x ∈ N and 2 < x < 4} = {3}.
Here between 2 and 3 only one element exists, hence B is called a Singleton Set.

Finite Set

A set that has a fixed or finite number of elements inside it is called a Finite Set.
For Example A = {x : x is an even number less than 10} then A = {2, 4, 6, 8}.
Here A has 4 elements, hence A is a finite set. The number of elements present in a finite set is called the Cardinal Number of a finite set. It is given as n(A) = 4.

Infinite Set

A set that has an indefinite or infinite number of elements inside it is called a Finite Set.
For Example A = {x : x is an even number l} then A = {2, 4, 6, 8……}. Here A has unlimited elements, hence A is an infinite set.

Equivalent Sets

If the number of elements present in two sets is equal i.e. the cardinal number of two finite sets is the same then they are called Equivalent Sets.
For Example, A = {x : x is an even number up to 10} = {2, 4, 6, 8, 10} and
B = {y : y is an odd number less than 10} = {1, 3, 5, 7, 9}.
Here, the cardinal number of set A is n(A) = 5 and that of B is given as n(B) = 5 then we see that n(A) = n(B). Hence A and B are equivalent sets.

Equal Sets

If the number of elements and also the elements of two sets are the same irrespective of the order then the two sets are called equal sets.
For Example, if set A = {2, 4, 6, 8} and B = {8, 4, 6, 2}
then we see that number of elements in both sets A and B is 4 i.e. same and the elements are also the same although the order is different. Hence, A and B are Equal Sets. Equal Sets are represented as A = B.

Unequal Sets

If at least any one element of one set differs from the elements of another set then the two sets are said to be unequal sets.

For Example, if set A = {2, 4, 6, 8} and B = {4, 6, 8, 10}

then set A and B are unequal sets as 2 is present in set A but not in B and 10 is present in set B but not in A. Hence, one element differs between them thus making them unequal. However, the cardinal number is the same therefore they are equivalent sets.

Overlapping Sets

If at least any one element of the two sets are the same then the two sets are said to be overlapping sets. For Example, if set A = {1, 2, 3} and set B = {3, 4, 5} then we see that 3 is the common element between set A and set B hence, set A and set B are Overlapping Sets.

Disjoint Sets

If none of the elements between two sets are common then they are called the Disjoint Sets i.e., for two sets A and B if A. For Example, set A = {1, 2, 3} and set B = {4, 5, 6} then we observe that there is no common element between set A and set B hence, set A and B are Disjoint Sets.

Apart from the above-mentioned sets, there are other sets called, Subsets, Supersets, Universal Sets, and Power Sets.

Subsets

If A and B are two sets such that every element of set A is present in set B then A is called the subset of B. It is represented as $A \subseteq B$ and read as 'A is a subset of B'. Mathematically it is expressed as

$$A \subseteq B \text{ iff}$$
$$a \in A \Rightarrow a \in B$$

If A is not a subset of B we write it as $A \not\subseteq B$.

For Example, if A = {1, 2} and B = {1, 2, 3} then we see that all the elements of A are present in B, hence $A \subseteq B$.

There are two kinds of subset Proper Subset and Improper Subset.

Proper Subset

If a subset doesn't contain all the elements of the set or <u>has fewer elements than the original set</u> then it is called the proper subset.

For example, in set A = {1, 2} and B = {1, 2, 3},

the subset A doesn't contain all the elements of the original set B, hence A is a proper subset of B. It is represented as $A \subset B$. Empty set is a proper subset of a given set as it has no elements.

Improper Subset

If a subset contains all the elements that are present in the original set then it is called an Improper Subset. For Example, if set A = {1, 2, 3, 4} and set B = {1, 2, 3, 4} then A is the improper subset of set B. It is mathematically expressed as $A \subseteq B$. Thus we deduce that two

sets are equal iff A ⊆ B and B ⊆ A. It should be noted that an empty set is an improper subset of itself.

Some Important Results on Subset
- ➢ Every set is a subset of itself
- ➢ An empty Set is a subset of every set.
- ➢ The number of possible subsets for a given finite set with 'n' number of elements is equal to 2n.
- ➢ N ⊂ W ⊂ Z ⊂ Q ⊂ R and T ⊂ R where N is a set of Natural Numbers, W is a set of Whole Numbers, Z is a set of integers, Q is a set of Rational Numbers, T is a set of irrational numbers and R is set of real numbers.

Superset

If all the elements of set A are present in set B then set B is called the Superset of set A. It is represented as B ⊇ A.

Let's say if A = {2, 3, 4} and B = {1, 2, 3, 4} then we see that all elements of set A are present in set B, hence B ⊇ A.

If a superset has more elements than its subset then it is called a proper or strict superset. A Proper Superset is represented as B ⊃ A.

Some of the Properties of Supersets are mentioned below:
- ➢ Every set is a superset of itself.
- ➢ Every set is a superset of an empty set.
- ➢ Total number of possible supersets for a given subset is infinite
- ➢ If B is a superset of A then A is a subset of B

Universal Set

The set that contains all the sets in it is called a Universal Set.
Let's say set A = {1, 2, 3}, set B = {4, 5}, and set C = {6, 7} then
Universal Set is given as U = {1, 2, 3, 4, 5, 6, 7}.
Another Example of a Universal Set is U = {Set of All Living Beings} then which includes both floras and faunas. Flora and fauna are the subsets of Universal Sets U.

Power Set

A set that contains all the subsets as its element is called the Power Set.
For Example, if set A = {1, 3, 5} then its subsets are {Φ}, {1}, {2}, {3}, {1, 3}, {3, 5}, {1, 5} and {1, 3, 5} then its Power Set is given as

P(A) = {{Φ}, {1}, {2}, {3}, {1, 3}, {3, 5}, {1, 5}, {1, 3, 5}}.
As we know the number of possible subsets for a finite set containing n elements is given by 2n then the number of elements in the power set is also 2n

Set Theory Symbols:

There are various symbols that are used in Sets Theory. The symbol and their explanation are tabulated below:

Symbol	Explanation
{}	Set
x ∈ A	x is an element of set A
x ∉ A	x is not an element of set A

∃ or ∄	There exist or there doesn't exist
Φ	Empty Set
A = B	Equal Sets
n(A)	Cardinal Number of Set A
P(A)	Power Set
A ⊆ B	A is a subset of B
A ⊂ B	A is the Proper subset of B
A ⊄ B	A is not a subset of B
B ⊇ A	B is the superset of A
B ⊃ A	B is a proper superset of A
B ⊅ A	B is not a superset of A
A ∪ B	A union B
A ∩ B	A intersection B
A'	Complement of Set A

Sets Operation/Combination of set

The sets undergo various operation which includes their union of sets, the intersection of sets, the difference of sets, the complement of sets, and the Cartesian Product of Sets. We will learn them briefly below:

Union of Sets

Union of Sets basically refers to uniting two sets and writing their elements in a single set without repeating elements if common elements are present.
The union of sets is given by A ∪ B.
For Example if Set A = {2, 4} and Set B = {4, 6} then
 A ∪ B = {2, 4} ∪ {4, 6} = {2, 4, 6}

Intersection of Sets

Intersection of sets refers to finding the common elements between two sets.
It is given by A ∩ B.
For Example if set A = {2, 4} and B = {4, 6} then
    A ∩ B = {2, 4} ∩ {4, 6} = {4}.

Difference of Sets

Difference of Sets refers to the deletion of common elements of two sets and writing down the remaining elements of two sets.
It is represented as A − B.
For Example if et A = {2, 4} and B = {4, 6} then
    A − B = {2, 6}

Complement of Set

Compliment of Set refers to the set of elements from the universal set excluding the elements of the set of which we are finding the compliment. It is given by A'.
For Example, if we have to find out the complement of the set of Natural Numbers then it will include all the numbers in the set from the Real Numbers except the Natural Numbers. Here Real Number is the Universal set of Natural Numbers.

Cartesian Product of Sets

Cartesian Product of Sets refers to the product between the elements of two sets in ordered pair. It is given as A ✕ B.
For Example if set A = {2, 4} and B = {4, 6} then
    A ✕ B = {(2,4), (2,6), (4,4), (4,6)}.

Example

If P = {5, 6}, form the set P × P × P.

Solution:

P × P × P = {5, 6} × {5, 6} × {5, 6}

= {(5, 5, 5), (5, 5, 6), (5, 6, 5), (5, 6, 6), (6, 5, 5), (6, 5, 6), (6, 6, 5), (6, 6, 6)}

Venn Diagram

Venn Diagram is a technique for representing the relation between two sets with the help of circles, generally intersecting.

For Example, two circles intersecting with each other with the common area merged into them represent the union of sets, and two intersecting circles with a common area highlighted represents the intersection of sets while two circles separated from each other represents the two disjoint sets. A rectangular box surrounding the circle represents the universal set. The Venn diagrams for various operations of sets are listed below:
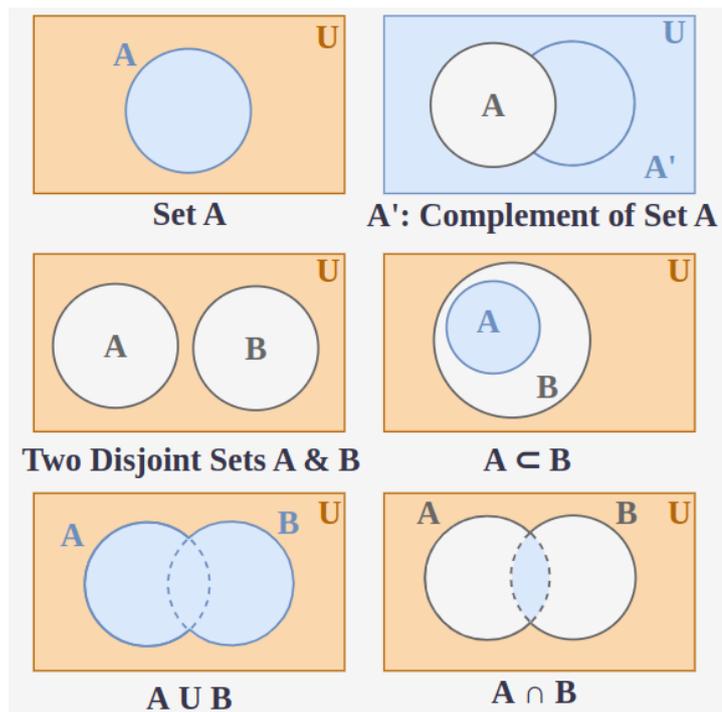
Examples on set

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 3\}$ and $D = \{7, 8, 9\}$.
Determine which of the following are true, false, or meaningless.

1. $A \subset B$.	2. $B \subset A$.	3. $B \in C$.
4. $\emptyset \in A$.	5. $\emptyset \subset A$.	6. $A < D$.
7. $3 \in C$.	8. $3 \subset C$.	9. $\{3\} \subset C$.

1. False. For example, $1 \in A$ but $1 \notin B$.

2. True. Every element in B is an element in A.

3. False. The elements in C are 1, 2, and 3. The *set* B is not equal to 1, 2, or 3.

4. False. A has exactly 6 elements, and none of them are the empty set.

5. True. Everything in the empty set (nothing) is also an element of A. Notice that the empty set is a subset of every set.

6. Meaningless. A set cannot be less than another set.

7. True. 3 is one of the elements of the set C.

8. Meaningless. 3 is not a set, so it cannot be a subset of another set.

9. True. 3 is the only element of the set $\{3\}$, and is an element of C, so every element in $\{3\}$ is an element of C.

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 3\}$ and $D = \{7, 8, 9\}$. If the universe is $\mathcal{U} = \{1, 2, \ldots, 10\}$, find:

1. $A \cup B$.	2. $A \cap B$.	3. $B \cap C$.
4. $A \cap D$.	5. $\overline{B \cup C}$.	6. $(D \cap \overline{C}) \cup \overline{A \cap B}$.
7. $\emptyset \cup C$.	8. $\emptyset \cap C$.	9.

1. $A \cup B = \{1, 2, 3, 4, 5, 6\} = A$ since everything in B is already in A.

2. $A \cap B = \{2, 4, 6\} = B$ since everything in B is in A.

3. $B \cap C = \{2\}$ as the only element of both B and C is 2.

4. $A \cap D = \emptyset$ since A and D have no common elements.

5. $\overline{B \cup C} = \{5, 7, 8, 9, 10\}$. First we find that $B \cup C = \{1, 2, 3, 4, 6\}$, then we take everything not in that set.

6. $(D \cap \overline{C}) \cup \overline{A \cap B} = \{1, 3, 5, 7, 8, 9, 10\}$. The set contains all elements that are either in D but not in C (i.e., $\{7, 8, 9\}$), or not in both A and B (i.e., $\{1, 3, 5, 7, 8, 9, 10\}$).

7. $\emptyset \cup C = C$ since nothing is added by the empty set.

8. $\emptyset \cap C = \emptyset$ since nothing can be both in a set and in the empty set.

Prepositional Logic

A proposition is a collection of declarative statements that has either a truth value "true" or a truth value "false". A propositional consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B, etc). The connectives connect the propositional variables.

Some examples of Propositions are given below −

- *"Man is Mortal", it returns truth value "TRUE"*
- *"12 + 9 = 3 − 2", it returns truth value "FALSE"*

The following is not a Proposition −

- *"A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true or false.*

Logical connectives

In propositional logic generally we use five connectives which are −
- OR (∨)
- AND (∧)
- Negation/ NOT (¬)
- Implication / if-then (→)
- If and only if (⇔)

OR (∨)

The OR operation of two propositions A and B (written as A∨B) is true if at least any of the propositional variable A or B is true.

The truth table is as follows −

A	B	A ∨ B
True	True	True
True	False	True
False	True	True
False	False	False

AND (∧)

The AND operation of two propositions A and B (written as A∧B) is true if both the propositional variable A and B is true.

The truth table is as follows −

A	B	A∧ B
True	True	True
True	False	False
False	True	False
False	False	False

Negation (¬)

The negation of a proposition A (written as ¬A) is false when A is true and is true when A is false.

The truth table is as follows −

A	¬ A
True	False
False	True

Implication / if-then (→)

An implication A→B is the proposition "if A, then B". It is false if A is true and B is false. The rest cases are true.

The truth table is as follows −

A	B	A → B
True	True	True
True	False	False
False	True	True
False	False	True

If and only if (⇔)

A⇔B is bi-conditional logical connective which is true when p and q are same, i.e. both are false or both are true.

The truth table is as follows −

A	B	A ⮂ B
True	True	True
True	False	False
False	True	False
False	False	True

Tautology

- A proposition P is a tautology if it is true under all circumstances.
- It is an assertion of Propositional Logic that is true in all situations; that is, it is true for all possible values of its variables
- The simple examples of tautology are;

  ➢ Either Mohan will go home or Mohan will not go home.
  ➢ He is healthy or he is not healthy
  ➢ A number is odd or a number is not odd.

Example: Prove that the statement (p→q) ↔(~q→~p) is a tautology.

Solution: Make the truth table of the above statement:

p	q	p→q	~q	~p	~q→~p	(p→q)↔(~q→~p)
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

As the final column contains all T's, so it is a tautology.

Contradiction

- A statement that is always false is known as a contradiction.
- It is an assertion of Propositional Logic that is false in all situations; that is, it is false for all possible values of its variables.

Example: Show that the statement p ∧~p is a contradiction.

Solution:

p	~p	p ∧~p
T	F	F
F	T	F

Since, the last column contains all F's, so it's a contradiction.

Normal Forms

A (P1, P2, ... , Pn) is a statements formula then the construction of the truth table may not be practical always. The problem of finding whether a given statement is tautology or contradiction or satisfiable in a finite number of steps is called the Decision Problem.

For Decision Problem, construction of truth table may not be practical always. We consider an alternate procedure known as the reduction to normal forms.

Some Important Law

(1) Idempotent law

$$p \wedge p \Leftrightarrow p \text{ and } p \vee p \Leftrightarrow p$$

(2) Commutative law

$$p \wedge q \Leftrightarrow q \wedge p \text{ and } p \vee q \Leftrightarrow q \vee p$$

(3) Associative law

$$(p \vee q) \vee r = p \vee (q \vee r)$$
$$\text{and } (p \wedge q) \wedge r = p \wedge (q \wedge r)$$

(4) De-Morgan law

$$\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$$
$$\text{and } \sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

(5) $p \rightarrow q \Leftrightarrow \sim p \vee q$

(6) $(\sim p \vee p) = 1, \quad (\sim p \wedge p) = 1, \quad \sim(\sim p) = p, \quad p \Leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

Types of Normal form

1. Disjunctive Normal form (DNF)

A statement form which consist of disjunction between conjunction is called DNF.

Examples:

$(p \wedge q) \vee r$

$(p \wedge \sim q) \vee (q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

Example : Obtain the DNF of the form $(p \rightarrow q) \wedge (\sim p \wedge q)$

Solution : We know that

$$p \rightarrow q \Leftrightarrow \sim p \vee q$$

So, $(\sim p \vee q) \wedge (\sim p \wedge q)$

Apply distributive law

$$(\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$$

$$\Rightarrow (\sim p \wedge q) \vee (q \wedge \sim p)$$

Obtain DNF of $p \lor (\sim p \to (q \lor (q \to \sim r)))$

Solution :

$\Leftrightarrow p \lor (\sim p \to (q \lor (\sim q \lor \sim r)))$

$\Leftrightarrow p \lor (p \lor (q \lor (\sim q \lor \sim r)))$

$\Leftrightarrow p \lor (p \lor (q \lor \sim q \lor q \lor \sim r)))$

$\Leftrightarrow p \lor (p \lor (q \lor \sim r))$

$\Leftrightarrow p \lor (p \lor q \lor p \lor \sim r)$

$\Leftrightarrow p \lor (p \lor q \lor \sim r)$

$\Leftrightarrow p \lor p \lor p \lor q \lor p \lor \sim r$

$\Leftrightarrow p \lor q \lor \sim r$

2. Conjunction Normal form (CNF)

A statement form which consist of conjunction between disjunction is called CNF.

Examples: $(\sim p \lor q) \land (\sim p \lor r)$

$(\sim p \lor q) \land (q \lor r) \land (\sim p \lor q \lor \sim r)$

Example: Find CNF of $p \land (p \to q)$

Solution : We know that $p \to q \Leftrightarrow \sim p \lor q$

$\Rightarrow p \land (\sim p \lor q) \Leftrightarrow (p \lor \sim p) \lor (p \land q)$

$\Rightarrow (p \land q)$

Example : Obtain CNF of the form $(p \land q) \lor (\sim p \land q \land r)$

Solution : $(p \land q) \lor (\sim p \land q \land r)$

Using distributive law

$(p \lor (\sim p \land q \land r)) \land (q \lor (\sim p \land q \land r))$

$[(p \lor \sim p) \land (p \lor q) \land (p \lor r)] \land [(q \lor \sim p) \land (q \lor q) \land (q \lor r)]$

$\Rightarrow (p \lor q) \land (p \lor r) \land [(q \lor \sim p) \land q \land (q \lor r)]$

Example:

Obtain CNF of $(p \rightarrow q) \wedge (q \vee (p \wedge r))$
and determine whether or not it is tautology.

Solution : We know that $p \rightarrow q \Leftrightarrow \sim p \vee q$
$$\Rightarrow (\sim p \vee q) \wedge (q \vee (p \wedge r))$$
$$\Rightarrow (\sim p \vee q) \wedge (q \vee p) \wedge (q \vee r)$$

Truth table

p	q	r	¬p	¬p∨q	q∨p	q∨r	(¬p∨q)∧(q∨p)∧(q∨r)
T	T	T	F	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	F	T	T	F
F	T	T	T	T	T	T	T
T	F	F	F	F	T	F	F
F	T	F	T	T	T	T	T
F	F	T	T	T	F	T	F
F	F	F	T	T	F	F	F

which is not tautology.

Q: Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.
a) ¬q   b) p ∧ q   c) ¬p ∨ q   d) p →¬q   e) ¬q → p f) ¬p →¬q   g) p ↔¬q   h) ¬p ∧ (p∨ ¬q)

a) The proposition ¬q as an English sentence is "Sharks have not been spotted near the shore"

b) The proposition p ∧ q   as an English sentence is "Swimming at the New Jersey shore is allowed and sharks have been spotted near the shore"

c) The proposition ¬p ∨ q   as an English sentence is "Swimming at the New Jersey shore is not allowed or sharks have been spotted near the shore"

d) The proposition p →¬q   as an English sentence is "If swimming at the New Jersey shore is allowed then sharks have not been spotted near the shore"

e) The proposition ¬q → p as an English sentence is "If sharks have not been spotted near the shore then swimming at the New Jersey shore is allowed"

f) The proposition ¬p →¬q as an English sentence is "If swimming at the New Jersey shore is not allowed then sharks have not been spotted near the shore"

g) The proposition p ↔¬q as an English sentence is "Swimming at the New Jersey shore is allowed if and only if sharks have not been spotted near the shore"

h) The proposition ¬p ∧ (p∨ ¬q) as an English sentence is "Swimming at the New Jersey shore is not allowed, and swimming at the New Jersey shore is allowed or sharks have not been spotted near the shore"

Predicate Logic

- Predicate is a sentence that contains one or more variables.
- A predicate becomes a proposition when the variables are substituted with specific values.

eg:

Ram is bachelor.

Sham is bachelor.

Then p(x): *x is bachelor.*

Where, x is **predicate variable** and p(x) **is propositional function.**

Examples:

Ex. Let $P(x): x > 3$. What are the truth values of $P(4)$ & $P(2)$.

Sol $P(4): 4 > 3$, which is true

$P(2): 2 > 3$, which is false

Ex. Let $Q(x, y): x = y + 3$. What are the truth values of the propositions $Q(1,2)$ & $Q(3,0)$?

Sol. $Q(1,2) \neq 1 = 2 + 3$, which is false.

$Q(3,0): 3 = 0 + 3$, which is true.

Ex. $P(x): x^2$ is a positive number for every real no. x.
Truth value?

sol. $P(x)$ is true for every x.

Ex. $P(x): x^2$ is a negative number for every real no. x.
Truth value?

Sol. $P(x)$ is false for every x.

Quantifiers

- The variable of predicates is quantified by quantifiers.
- Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.
- Quantifiers allow us to talk about all objects or the existence of some object

There are two quantifiers in first-order logic:

1. Universal quantifier

2. Existential quantifier

Quantifiers

Quantifiers are words that refer to quantities such as some, few, many, all, none and indicate how frequently a certain statement is true.

Types of Quantifier

Universal Quantifier	Existential Quantifier
1. The phrase "for all" is called universal quantifier.	1. The phrase "there exists" is called Existential quantifier.
2. It is denoted by $\lor$, $\forall$	2. It is denoted by $\exists$.
3. $\forall x$ represents "for all x" "for every x" "for each x"	3. $\exists x$ represents "there exists an x" "there is an x" "for some x" "there is at least one x"
4. All human beings are mortal. —① P(x): x is mortal ① can be written as $(\forall x \in U) \, P(x)$ or $\forall x \, P(x)$ U — domain or universe of discourse.	4. Some human beings are mortal P(x): x is mortal $\exists x \, P(x)$
5. All students are successful. P(x): x is successful. $\forall x \, P(x)$	5. Some students are successful P(x): x is successful $\exists x \, P(x)$

Translate sentences into logical expressions

1. Every person is precious.
   Rephrase: For every x, if x is a person then x is precious
   M(x): x is a person,   A(x): x is precious
   ∴ ∀x(M(x) → A(x))

2. Every student is clever.
   M(x): x is a student ;   N(x): x is clever
   ∀x (M(x) → N(x)).

3. All men are mortal
   M(x): x is a man , P(x): x is mortal
   ∀x (M(x) → P(x)).

4. Any integer is either positive or negative
   M(x): x is an integer   P(x): x is either positive or negative
   ∀x (M(x) → P(x))

5. There exists a student.
   P(x): x is a student.   ∃x(P(x))

6. Some students are clever.
   P(x): x is a student.   Q(x): x is clever.
   ∃x (P(x) ∧ Q(x))

7. Some students are not successful.
   P(x): x is a student.   Q(x): x is successful
   ∃x (P(x) ∧ ~Q(x))

9. Not all birds can fly
   P(x): x is a bird
   Q(x): x can fly   ~ [∀x (P(x) → Q(x))]

Q: Rewrite the following using quantifiers, variables &
   predicate symbols.
   ⓐ All birds can fly.   Ans. ∀x[B(x) → F(x)]
   ⓑ Some men are genius.   Ans. ∃x[M(x) ∧ G(x)]
   ⓒ Some numbers are not rational   Ans ∃x[N(x) ∧ ~R(x)]
   ⓓ All mammals are animals   Ans. ∀x [M(x) → A(x)]
   ⓔ No natural no. is negative.   Ans. ~[∀x (N(x) → m(x))]

Logical Equivalences involving Quantifiers

Two logical statements involving predicates and quantifiers are considered equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements irrespective of the domain used for the variables in the propositions.

Negating Quantified statements

Consider the statement "Every Computer Science Graduate has taken a course in Discrete Mathematics."

The above statement is a universal quantification, $xP(x)$

where $P(x)$ is the statement "x has taken a course in Discrete Mathematics" and the domain of x is all Computer Science Graduates.

The negation of this statement is "It is not the case that every computer science graduate has taken a course in Discrete Mathematics" or simply "There is a computer science graduate who has not taken a course in Discrete Mathematics".

The above statement can be expressed using an existential quantification.

$\exists x \neg P(x)$

Thus, we get the following logical equivalence-

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

Similarly,

$\neg \exists x P(x) \equiv \forall x \neg P(x)$

These equivalences are nothing but rules for negations of quantifiers. They are also known as **De Morgans's laws for quantifiers.**

Q: Transcribe the following into logical notation. Let the universe of discourse be the real numbers.

i. For any value of x, x2 is non-negative.

$$(\forall x \in R)P(x^2 > 0)$$

ii. For every value of x, there is some value of y such that $x.y = 1$.

$$(\forall x \in R \ \exists y \in R) \ P(x.y = 0)$$

iii. There are positive values of x and y such that $x.y > 0$.

$$(\exists x \ \exists y) \ P((x > 0)(y > 0) \rightarrow (x.y > 0))$$

iv. For every real numbers x and y, if x is positive and y is negative then xy is negative.

$$(\forall x \ \forall y) \ P((x > 0)(y < 0) \rightarrow (xy < 0))$$

v. There is a value of x such that if y is positive, then x + y is negative.

$$(\exists x \ \exists y) \ P((y > 0) \rightarrow (x + y < 0))$$

Q Translate the following statement into English.

$$\forall x \, (C(x) \lor \exists y \, (C(y) \land F(x,y)))$$

C(x): x has a computer.
F(x,y): x and y are friends.
Domain of x and y: all students

Solution:

"For every student x, x has a computer or there is a student y such that y has a computer and x and y are friends."

"Every student has a computer or has a friend that has a computer."

Principle of mathematical induction

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of n, where n is a positive integer.

Let P(n) be a given statement involving the natural number n such that

(i)   The statement is true for n = 1, i.e., P(1) is true (or true for any fixed natural number) and
(ii) If the statement is true for n = k (where k is a particular but arbitrary natural number), then the statement is also true for n = k + 1, i.e, truth of P(k) implies the truth of P(k + 1). Then P(n) is true for all natural numbers n.

Example: Prove statement by using the Principle of Mathematical Induction for all n ∈ N, that

$$1 + 3 + 5 + \ldots + (2n - 1) = n2$$

Solution:

Let the given statement P(n) be defined as

P(n) : $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, for n ∈ N.

Note that P(1) is true, since

P(1) : $1 = 1^2$

Assume that P(k) is true for some k ∈ N, i.e.,

P(k) : $1 + 3 + 5 + \ldots + (2k - 1) = k^2$

Now, to prove that P(k + 1) is true, we have

$1 + 3 + 5 + \ldots + (2k - 1) + (2[k + 1]-1) = k^2 + (2k + 1)$

$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + 2k + 1$

$$= (k + 1)^2$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all n ∈ N.

Example Show that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad n \geq 1$$

Solution:

Let the given statement P(n) be defined as

$$P(n) = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad n \geq 1$$

by mathematical induction.

1. *Basis of induction.* For $n = 1$, we have

$$1^2 = \frac{1(1+1)(2+1)}{6}$$

P(1) is true

2. *Induction step.* Assume that P(k) is true for some $k \in N$, i.e.,

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now, to prove that P(k + 1) is true, We have

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all $n \in N$.

Well-ordering principle

The well-ordering principle states that every non-empty set of positive integers has a least element. This means that given any set of positive integers, there is always a smallest number in that set.

To understand this principle better, let's look at an example. Consider the set of positive integers {1, 2, 3, 4, ...}. This set is well-ordered because it has a smallest element, which is 1. Every other element in the set is greater than 1, so 1 is the least element.

Now let's consider the set of positive even integers {2, 4, 6, 8, ...}. This set is also well-ordered because it has a smallest element, which is 2. Every other element in the set is greater than 2, so 2 is the least element.

Proof of the Well-Ordering Principle

Well-Ordering Principle: Every non-empty subset S of the set of positive integers $\mathbb{N}$ contains a least element. That is, there exists an element $m \in S$ such that $m \leq n$ for all $n \in S$.

Proof: Let S be a non-empty subset of $/\mathbb{N}$. We will prove the Well-Ordering Principle by contradiction. Suppose that S does not have a least element, that is, for any $m \in S$, there exists an $n \in S$ such that $n < m$.

Now, consider the set $T = k \in \mathbb{N} \mid m \in S \mid m < k$ **is non-empty.** Intuitively, T is the set of all positive integers k such that the set of elements in S that are less than k is non-empty.

Since S is non-empty, there exists some m\in S. Therefore, $m \in S \mid m < m$ is empty, which implies that $1 \in$ T.

Now, for each $n \in \mathbb{N}$, define m_n to be the least element of the set $m \in S \mid m \geq n$. Since S does not have a least element, we have $m_n < n$ for all $n \in \mathbb{N}$. Therefore, the sequence (m_n) is a decreasing sequence of positive integers.

But this contradicts the fact that every decreasing sequence of positive integers must have a lower bound. Specifically, since $m_n \in S$ for all n, the set S contains an infinite decreasing sequence of positive integers, which contradicts the fact that $\mathbb{N}$ is well-ordered.

Therefore, our initial assumption that S does not have a least element must be false, and so the Well-Ordering Principle holds.

Recursive definition

Recursive Function is a function that repeats or uses its own previous term to calculate subsequent terms and thus forms a sequence of terms.

For any recursively defined function, it has two parts. The first part is the definition of the smallest argument and the second part is the definition of the nth term. The smallest argument is usually denoted by f(0) or f(1) and the nth argument is denoted by f(n).

Let the sequence be 5, 7, 9, 11

The recursive formula for the given sequence is given by

f(0)= 5

f(n) = f(n-1)+2

Now, we can check the sequence terms using the recursive formula as follows:

f(0)= 5

f(1) = f(0)+2

f(1)= 5+2 = 7

f(2) = f(1)+2

f(2)= 7+2 =9

f(3)= f(2)+2

f(3)= 9+2 = 11

In this way, we can find the next term in the sequence with the help of the recursive function formula.

The Division Algorithm

If *a* and *d* are any integers with *d* not zero, then there are unique integers *q* and *r* such that

$$a = dq + r \text{ with } 0 \leq r < |d|.$$

For example, if *a* is 36 and *d* is 13, then *q* = 2 and *r* = 10 (since 36 = 2·13 + 10).

Likewise if *a* is -63 and *d* is 20, then *q* = -4 and *r* = 17 (since -63 = -4·20 + 17).

Finally, if *a* is 24 and *d* is -15, then *q* = -1 and *r* = 9 (since 24 = -1·(-15) + 9).

The unique numbers *q* and *r* are called the **quotient** and **remainder** respectively. The remainder is also called the least nonnegative residue modulo *d*.

Observations:

- *In the Division Algorithm it is required that 'd', the divisor, be a positive integer.*
- *If the remainder is equal to zero then the division of 'a' by 'd' is exact (or equivalently 'a' divides evenly into d). In this case both d and q are factors of 'd'.*
- *If the remainder is different than zero then there must be a remainder (r) that must $0 \leq r < d$. In this case, 'a' does not divide evenly into d. Therefore, d is not a factor of 'a'.*
- *As a consequence of the previous statement any statement of the form 'a' = d*q + 1 indicates that d does not divide 'a'.*

Prime Numbers

Let p be an integer such that p>1. We say that p is a *prime number*, or simply p is *prime*, if the only positive numbers that divide p are 1 and p itself. An integer n>1 that is not prime is said to be *composite*.

Greatest Common Divisor (GCD)

- Suppose a & b are integers and not both are zero. An integer d is called a common devisor of a & b if d divides both a & b.
- 1 is common devisor of both a & b.
- Any common devisor of both a & b can't be greater than both a & b.
- It is denoted by gcd(a,b).
  Eg 1. gcd(12,18)
    The common devisor are:
    $$\pm 1, \pm 2, \pm 3, \pm 6$$
    The greatest common devisor is(+6), therefore gcd(12,18)=6
  Eg 2. gcd(12,-18)=6

Euclidean algorithms

- The Euclidean algorithm is a way to find the greatest common divisor of two positive integers.
- It is extension of GCD.
- Let 'a, & 'b' are integers and d= gcd(a,b), then

$$d = ax + by$$

- It is calculated by using repeated division, means we repeatedly do the division until remainder is zero.

Example:

Let $a = 540$, $b = 168$

First we devide repeatedly, till we getting remainder as zero.

$$
\begin{array}{r}
3 \\
168 \overline{)\ 540} \\
-504 \\
\hline
36
\end{array}
\qquad
\begin{array}{r}
4 \\
36 \overline{)\ 168} \\
-144 \\
\hline
24
\end{array}
\qquad
\begin{array}{r}
1 \\
24 \overline{)\ 36} \\
-24 \\
\hline
12
\end{array}
\qquad
\begin{array}{r}
2 \\
12 \overline{)\ 24} \\
-24 \\
\hline
00
\end{array}
$$

i.e $\gcd(540, 168) = \gcd(168, 36) = \gcd(36, 24) =$

$$\gcd(24, 12) = \underline{12}$$

Now we have to apply Euclidean algorithm

$$d = ax + by$$

$$12 = 540x + 168y$$

1) $540 = 3(168) + 36$ or $36 = 540 - 3(168)$ --- ①

2) $168 = 4(36) + 24$ or $24 = 168 - 4(36)$ --- ②

3) $36 = 1(24) + 12$ or $12 = 36 - 1(24)$ --- ③

In eqⁿ ③ 12 is linear combination of 24 & 36

$$
\begin{aligned}
12 &= 36 - 1(24) \\
&= 36 - 1(168 - 4(36)) \\
&= 36 - 1(168) + 4(36) \\
&= 5(36) - 1(168) \\
&= 5(540 - 3(168)) - 1(168) \quad \cdots \text{we eqⁿ ①} \\
&= 5(540) - 15(168) - 1(168) \\
&= 5(540) - 16(168)
\end{aligned}
$$

$$\therefore x = 5 \quad y = -16$$

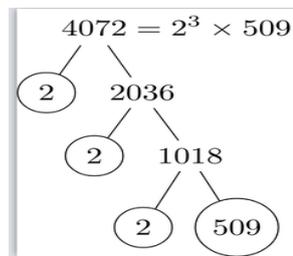# The Fundamental Theorem of Arithmetic

Every composite number can be expressed (factorized) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur. This is called **Fundamental Theorem of Arithmetic.**

This theorem says that every composite number can be rewritten as the product of prime numbers in a "unique" way, except for the order in which primes occur.
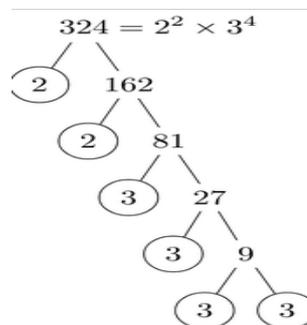
Fundamental Theorem of Arithmetic states that every integer greater than 1 is either a prime number or can be expressed in the form of primes. In other words, all the natural numbers can be expressed in the form of the product of its prime factors. The prime factors are the numbers which are divisible by 1 and itself only.

Therefore, **every natural number can be expressed in the form of the product of the power of its primes**. This statement is known as the **Fundamental Theorem of Arithmetic, unique factorization theorem or the unique-prime-factorization theorem**.
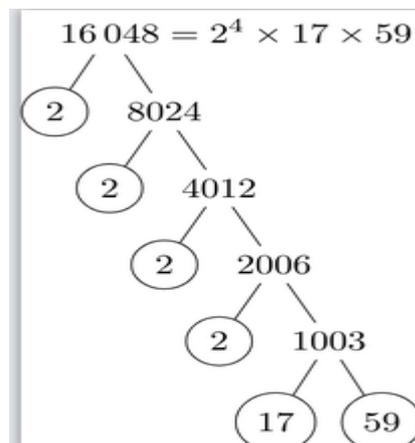
Example: **Factorize the number "4072"**

$$4072 = 2^3 \times 509$$

2   2036

2   1018

2   509

Factorize the number "324"

$$324 = 2^2 \times 3^4$$

2   162

2   81

3   27

3   9

3   3

Factorize the number "16048"

$$16\,048 = 2^4 \times 17 \times 59$$

2   8024

2   4012

2   2006

2   1003

17   59

[UNIT 2] Functions and Relations

PRODUCT SETS

Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where a ∈ A and b ∈ B is called the product, or Cartesian product, of A and B. A short designation of this product is A × B, which is read "A cross B." By definition,

Let $A = \{a, b\}$ and $B = \{a, c, d\}$. Then

$A \times B = \{(a, a), (a, c), (a, d), (b, a), (b, c), (b, d)\}$

Relation

In discrete mathematics, the relation can be described as a collection of ordered pairs. Let A and B be sets. A binary relation or, simply, relation from A to B is a subset of A × B. A binary relation indicates some elements in A are related to some elements to B.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B.
For example let A={a,b,c,d} be the set of four students, let B={CS301,CS303,CS305,CS309,CS310} be a set of five courses. The Cartesian product AxB gives all the possible pairings of students and courses. On the other hand relation

R={(a,CS301), (b,CS305), (b,CS310), (c,CS301), (d,CS303)}might describe the courses student taking admission.

Representing binary relations

Besides a list of the ordered pairs, a binary relation can also be represented in tabular form or graphical form. For example, let $A = \{a, b, c, d\}$ and $B = \{\alpha, \beta, \gamma\}$, and let $R = \{(a, \alpha), (b, \gamma), (c, \alpha), (c, \gamma), (d, \beta)\}$ be a binary relation from A to B.

R can be represented in tabular form, as shown in Fig. a, where the rows of the table correspond to the elements in A and the columns of the table correspond to the elements in B, and a check mark in a cell means the element in the row containing the cell is related to the element in the column containing the cell.

R can also be represented in graphical form as shown in Fig. b, where the points in the left-hand column are the elements in A, the points in the right-hand column are the elements in B, and an arrow from a point in the left-hand column to a point in the right-hand column indicates that the corresponding element in A is related to the corresponding element in B.

	α	β	γ
a	✓		
b			✓
c	✓		✓
d		✓	

(a)
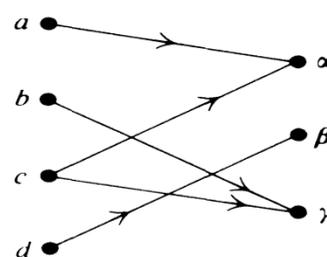
(b)

Since binary relations are sets of ordered pairs, the notions of the intersection of two relations, the union of two relations, the symmetric difference of two relations, and the difference of two relations follow directly from that of sets. let R_1 and R_2 be two binary relations from A to B. Then $R_1 \cap R_2$, $R_1 \cup R_2$, $R_1 \oplus R_2$, and $R_1 - R_2$ are also binary relations from A to B, which are known as the intersection, the union, the symmetric difference, and the difference of R_1 and R_2.

For example, let $A = \{a, b, c, d\}$ be a set of students and $B = \{CS121, CS221, CS257, CS264, CS273, CS273, CS281\}$ be a set of courses. We might have a binary relation R_1 from A to B describing the courses the students are taking, and a binary relation R_2 from A to B describing the courses the students are interested in, as shown in Fig. 4.2.

	CS121	CS221	CS257	CS264	CS273	CS281
a	✓					
b		✓	✓			
c		✓			✓	✓
d				✓		✓

R_1

	CS121	CS221	CS257	CS264	CS273	CS281
a	✓			✓		
b		✓			✓	
c						
d				✓	✓	✓

R_2

Then the binary relations $R_1 \cap R_2$, which is

$\{(a, CS121), (b, CS221), (d, CS264), (d, CS281)\}$, describes the courses that the students are taking and are also interested in. The binary relation $R_1 \cup R_2$, which is $\{(a, CS121), (a, CS264), (b, CS221), (b, CS257), (b, CS273), (c, CS221), (c, CS273), (c, CS281), (d, CS264), (d, CS273), (d, CS281)\}$, describes the courses the students are either taking or interested in. The binary relation $R_1 \oplus R_2$, which is $\{(a, CS264), (b, CS257), (b, CS273), (c, CS221), (c, CS273), (c, CS281), (d, CS273)\}$, describes the courses the students are interested in but not taking or are taking but not interested in. The binary relation $R_1 - R_2$, which is $\{(b, CS257), (c, CS221), (c, CS273), (c, CS281)\}$, describes the courses the students are taking but not interested in.

Properties of Binary Relations

Reflexive
- A relation is a reflexive relation if every element of set A maps to itself.
- i.e for every a ∈ A, (a, a) ∈ R. A relation R on set A is called Reflexive if ∀a∈A is related to a (**aRa holds**)

Example1 − The relation R={(a,a),(b,b)} on set X={a,b} is reflexive.

Example 2: Assume relation R={(a b), if a |b} on A = {1,2,3,4} Is R reflexive?

R= {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

•Answer: Yes. Integer always divide itself, therefore R is reflexive relation (1,1), (2,2), (3,3), and (4,4) ∈A.

Irreflexive

- A relation R on set A is called Irreflexive if no a∈A is related to a (**aRa does not hold**).

Example − The relation R={(a,b),(b,a)}on set X={a,b} is irreflexive.

relation on a set is represented in tabular form, it is very simple to determine whether the binary relation is a reflexive relation. To be specific, a binary relation on a set is reflexive if and only if all the cells on the main diagonal of the table contain check marks. For example, the binary relation in Fig. *a* is reflexive while that in Fig. *b* is not.

	a	b	c	d
a	✓			✓
b		✓		✓
c			✓	
d				✓

(a)

	a	b	c	d
a	✓			✓
b		✓		✓
c				
d				✓

(b)

Symmetric

- A relation R on set A is called Symmetric if xRy implies yRx, ∀x∈A and ∀y∈A.
- Suppose there are two elements in a set A, i.e., a, b. The relation R on a set A will be called symmetric relation if element 'a' has relation with 'b' is true, then 'b' has a relation with 'a' will also be true. The symmetric relation is described as follows:

$$\forall\ a, b \in A\ \ (a,b) \in R \rightarrow (b,a) \in R.$$

Example − The relation R={(1,2),(2,1),(3,2),(2,3)}on set A={1,2,3} is symmetric.

When a binary relation on a set is represented in tabular form, we can determine whether it is symmetric by observing whether the check marks are in cells that are symmetrical with respect to the main diagonal. For example, the binary relation in Fig. *a* is symmetric while that in Fig. *b* is not.

	a	b	c	d
a		✓		✓
b	✓		✓	
c		✓		✓
d	✓		✓	✓

(a)

	a	b	c	d
a		✓		✓
b	✓			
c				
d				✓

(b)

Anti-Symmetric

Let R be a binary relation on A. R is said to be an *antisymmetric relation* if (a, b) in R implies that (b, a) is not in R unless $a = b$. In other words, if both (a, b) and (b, a) are in R, then it must be the case that $a = b$.

For example, let A be a set of tests to be performed on a patient in the hospital and let R be a binary relation on A such that

if (a, b) is in R, then test a must be performed before test b.
Clearly, if test a must be performed before test b, then test b must not be performed before test a for two distinct tests a and b.
Thus, R is an antisymmetric relation.

As another example, let A be a set of positive integers and R be a binary relation on A such that

$R=\{(a,b)\rightarrow N|a\leq b\}$ (a, b) is in R if and only if $a \geqq b$.

We note that R is an antisymmetric relation.

Antisymmetric Relation Examples

Example 1: Check whether the given relationship is Antisymmetric or not.
$R = \{(a, a), (b, b), (c, c)\}$ on $X = \{a, b, c\}$

Solution: In the given relation R we see that all the ordered pairs have dissimilar elements. Therefore R satisfies the following condition, $(a, b) \in R$ and $(b, a) \in R$,

Thus we can say that a=b and hence the given relation is an Antisymmetric Relation.

Example 2: If $S = \{1, 2, 3, 4, 5\}$, then find the Antisymmetric Relation for set S.

Solution: We have to find such a relation that satisfies the following condition for becoming Antisymmetric.

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$

With this condition we get,

$R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

Thus R is the resultant Antisymmetric Relation.

Transitive

- A relation R on set A is called Transitive if xRy and yRz implies that xRz, where $\forall x,y,z \in A$
- Suppose there are three elements in a set A, i.e., a, b, and c. The relation R on a set A will be known as transitive relation if 'a' has a relation with 'b' and 'b' has a

relation with 'c', then 'a' will be also has a relation with 'c'. The transitive relation is described as follows:

If (a, b) ∈ R , (b, c) ∈ R , then (a, c) ∈ R for all a, b, c ∈ A

Example 1: The relation R={(1,2),(2,3),(1,3)} on set A={1,2,3} is transitive.

Example 2: Let A = { 3, 4, 5 } and R be a relation defined on set A as "is less than" and R = {(3, 4), (4, 5), (3, 5)}. Verify that R is transitive in nature.

> *Solution:* From the given set A, let:a = 3 b = 4   c = 5
> Here, we have;
> (a, b) = (3, 4) -> this implies 3 is less than 4.
> (b, c) = (4, 5) -> this implies 4 is less than 5.
> (a, c) = (3, 5) -> this implies 3 is less than 5.
> That is, if 3 is less than 4 and 4 is less than 5, then 3 is less than 5.
> More precisely in terms of mathematical notation,
> 3R4, 4R5 -> 3R5
> The above explanation proves that R is transitive.

Transitive closure

Let R be a binary relation on A. The *transitive extension of R*, denoted R_1, is a binary relation on A such that R_1 contains R, and moreover, if (a, b) and (b, c) are in R, then (a, c) is in R_1.

For example, let $A = \{a, b, c, d\}$ and R be the binary relation shown in Fig. a.

The transitive extension of R, is R_1, shown in Fig. b, where the ordered pairs in R_1 but not in R are marked with heavy check marks.

Let R_2 denote the transitive extension of R_1, and, in general, let R_{i+1} denote the transitive extension of R_i. We define the *transitive closure* of R, denoted R^*, to be the set union of R, R_1, R_2, ... For example, the transitive closure of the binary relation R in Fig. a is shown in Fig. c.

Example: Let R = {(a, b), (b, c), (c, a)} on set A={a,b,c} find transitive closure.

R*={(a, b),(b, c),(c, a),(a, c),(b, a),(c ,b),(a, b),(b, b),(c, c)}

Example: Let R = {(1,2), (2,3), (3,4),(2,1)} on set A={1,2,3,4} find transitive closure.

R* = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3,4)}

Equivalence Relation

- A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive.
- If (a, b) ∈ R , (b, c) ∈ R , then (a, c) ∈ R for all a, b, c ∈ A

Example – The relation R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)} on set A={1,2,3} is an equivalence relation since it is reflexive, symmetric, and transitive.

Problem: Let x = {1, 2, 3, ... 7} and R = {(x, y) / x – y is divisible by 3} Show that R is an equivalence relation.

Solution: For any a ∈ X, a- a is divisible by 3, Hence a R a, R is reflexive

For any a, b ∈ X, if a – b is divisible by 3, then b – a is also divisible by 3, R is symmetric.

For any a, b, c ∈ X, if aRb and bRc, then a – b is divisible by 3 and b–c is divisible by 3. So that (a – b) + (b – c) is also divisible by 3, hence a – c is also divisible by 3. Thus R is transitive.

Hence R is equivalence.

Problem: *Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} is R = { (a, b) : b = a+1 } is reflexive, symmetric or transitive. Justify your answer.*

Let A = {1, 2, 3, 4, 5, 6}.

A relation R is defined on set A as:

R = {(a, b): b = a + 1}

∴R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

We can find (a, a) ∉ R, where a ∈ A.

For instance,

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) ∉ R

∴R is not reflexive.

It can be observed that (1, 2) ∈ R, but (2, 1) ∉ R.

∴R is not symmetric.

Now, (1, 2), (2, 3) ∈ **R**

But,

(1, 3) ∉ R

∴R is not transitive

Hence, R is neither reflexive, nor symmetric, nor transitive.

Functions

- Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a function from A into B.
- The set A is called the domain of the function, and the set B is called the range or codomain.
- For each element a∈A, we associate a unique element b∈B. The set of all such associations is called a function f from A into B, denoted
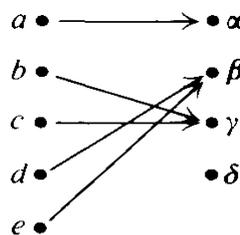
$$f:A\rightarrow B$$

for a function f from A to B instead of writing $(a,b) \in f$ we also use the notation:

$$f:a\rightarrow b, \text{ or } f(a)=b$$

Where b is called **_image_** of a. The set A is called **_domain_** of function f, and set B is called **_range_** of function f.

Representation of function:

Function can be represented in graphical form. Following figure shows the function from $A = \{a, b, c, d, e\}$ to $B = \{\alpha, \beta, \gamma, \delta\}$.
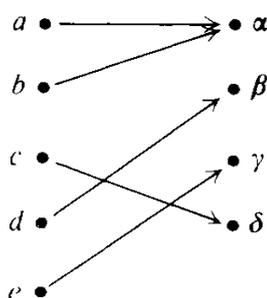

Types of Functions

- ## Surjective function

A function from A to B is said to be an *onto* function if every element of B is the image of one or more elements of A.
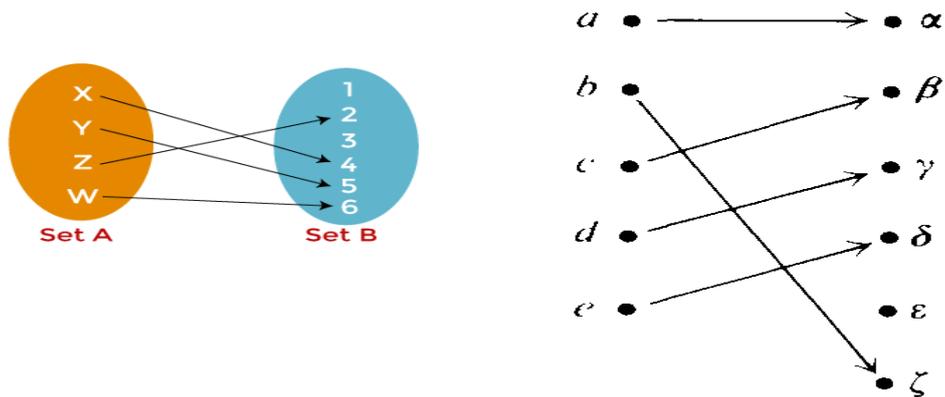
The *onto* function is also called *surjection*. Following figure shows the example of *onto* function.

- **Injective function**

A function from *A* to *B* is said to be a *one-to-one* function if no two elements of *A* have the same image.
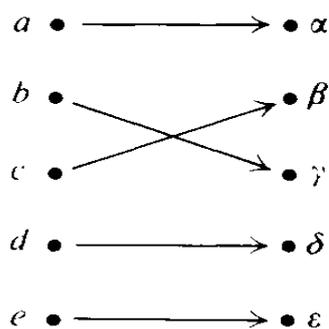
The *one-to-one* function is also called *injection*. Following figure shows the example of *one-to-one* function.

- **Bijective function**

A function from *A* to *B* is said to be a *one-to-one onto* function if it is both an onto and a one-to-one function.

The *one-to-one onto* function is also called *bijection*. Following figure shows the example of *one-to-one onto* function.
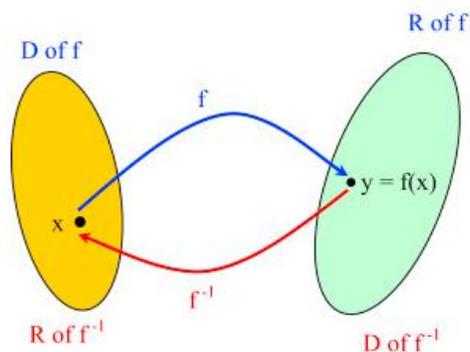

- **Inverse of function f−1**

Let f be a **one-to-one** function with domain D and range R. Then its inverse f^{-1} has domain R and range D, that is,

$$f(x) = y \Leftrightarrow f^{-1}(y) = x, \text{ for any } y \text{ in R and } x \text{ in D.}$$

The below figure illustrates the relation between the function f and its inverse f^{-1}.

Example 1

Determine whether each of the following functions is invertible. If it is invertible, find its inverse.

$f = \{(0, 2), (3, 2), (5, 6), (7, 13)\}$

$g = \{(-4, 7), (5, 9), (8, 15), (10, 19)\}$

Solution

The function f is not one-to-one because of the ordered pairs (0, 2) and (3, 2), therefore f is not invertible.

The function g is one-to-one, therefore g is invertible,

where $g^{-1} = \{(7, -4), (9, 5), (15, 8), (19, 10)\}$.

Composition of a function

- The composition of a function is an operation where two functions say f and g generate a new function say h in such a way that $h(x) = g(f(x))$.
- It means here function g is applied to the function of x. So, basically, a function is applied to the result of another function.
- Let $f: A \rightarrow B$ and $g : B \rightarrow C$ be two functions. Then the composition of f and g, denoted by $g \circ f$, is defined as the function $g \circ f : A \rightarrow C$ given by
  $$(g \circ f)(x) = g(f(x)), \ \forall \ x \in A.$$

The below figure shows the representation of composite functions.
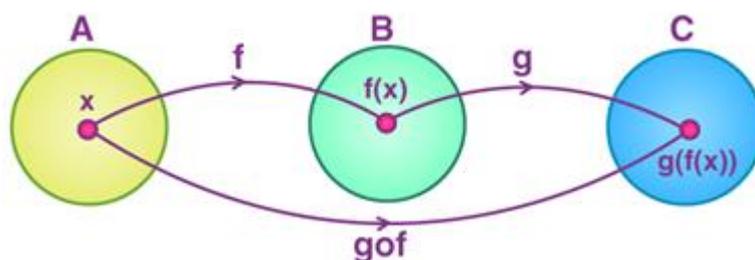
Example: If, f(x) = x² and g(x) = x + 3. Then calculate the composition g(f(x)) and f(g(x)).

$g(f(x)) = g(x^2) = x^2 + 3$

Similarly, we can easily calculate the f(g(x)) or (f∘g)(x) where g(x) is computed first.

$f(g(x)) = f(x + 3) = (x + 3)^2$

Example: If f(x) = x + 2 and g(x) = (2x)² find, f(g(x)) and g(f(x))
Solution:

For f(g(x))

Given, $g(x) = (2x)^2 = 4x^2$
Thus, $f(g(x)) = f(4x^2) = 4x^2 + 2$

Now for g(f(x))

Given, $f(x) = x + 2$

Thus, $g(f(x)) = (2(x+2))^2 = (2x + 4)^2$
$\Rightarrow g(f(x)) = 4x^2 + 16 + 16x$

Combinatorics

- Combinatorics is the branch of Mathematics dealing with the study of finite or countable discrete structures.
- It includes the counting of objects having certain properties.
- Counting helps us solve several types of problems such as counting the number of available IPv4 or IPv6 addresses.

Counting Principles: There are two basic counting principles, sum rule, and the product rule.

- **Sum Rule** – If a task can be done in one of n_1 ways or one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.
- **Product Rule** – If a task can be broken down into a sequence of k subtasks, where each subtask can be performed in n_1, n_2,.. n_k respectively, then the total number of ways the task can be performed is $n_1*n_2*.. *n_k$

Example 1 – In how many ways can a person choose a project from three lists of projects of sizes 10, 15, and 19 respectively?

Solution – The person has a choice of choosing a project from either of the three lists. So the person can choose from either 10 projects or 15 projects or 19 projects. Since choosing from one list is not the same as choosing another list, the total number of ways of choosing a project by the sum-rule is 10 + 15 + 19 = 44.

Example 2 – How many distinct license plates are possible in the given format- Two alphabets in uppercase, followed by two digits then a hyphen, and finally four digits. Sample- AB12-3456.

Solution – There are 26 possibilities for each of the two letters and 10 possibilities for each of the digits. Therefore the total number of possibilities is – 26 * 26 * 10 * 10 * 10 * 10 * 10 * 10 = 676000000.

Recurrence relation

- Recursive techniques can derive sequences and be used for solving counting problems. The procedure for finding the terms of a sequence in a recursive manner is called **recurrence relation**.
- A recurrence relation is an equation that recursively defines a sequence, where the next term is a function of the previous terms.
- **Example** − Fibonacci series $F_n = F_{n-1} + F_{n-2}...$

Linear Recurrence Relations

- A linear recurrence equation of degree k or order k is a recurrence equation which is in the format

$$x_n = A_1 x_{n-1} + A_2 x_{n-2} + A_3 x_{n-3} + ... A_k x_{n-k}$$

  (A_n is a constant and $A_k \neq 0$) on a sequence of numbers as a first-degree polynomial.

Generating Functions

- Generating Functions represents sequences where each term of a sequence is expressed as a coefficient of a variable x in a formal power series.
- Mathematically, for an infinite sequence, say a0,a1,a2,…,ak,…, the generating function will be −

$$G_x = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Some Areas of Application

Generating functions can be used for the following purposes −

- For solving a variety of counting problems. For example, the number of ways to make change for a Rs. 100 note with the notes of denominations Rs.1, Rs.2, Rs.5, Rs.10, Rs.20 and Rs.50
- For solving recurrence relations
- For proving some of the combinatorial identities
- For finding asymptotic formulae for terms of sequences

Problem

What are the generating functions for the sequences $\{a_k\}$ with $a_k = 2$ and $a_k = 3k$?

Solution

When $a_k = 2$, generating function,

$G(x) = \sum_{k=0}^{\infty} 2x^k = 2 + 2x + 2x^2 + 2x^3 + \ldots$

When $a_k = 3k$, $G(x) = \sum_{k=0}^{\infty} 3kx^k = 0 + 3x + 6x^2 + 9x^3 + \ldots\ldots$